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The Footballers of Croam

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The vertices of the graph O_k are indexed by the $(k-1)$-subsets of a $(2k-1)$-set. Two vertices are adjacent if and only if their labelling sets are disjoint. This paper establishes that O_5 is an edge disjoint union of two Hamiltonian circuits and a 1-factor and that O_6 is an edge disjoint union of three Hamiltonian circuits. It follows that the graphs are 5, 6-edge chromatic, respectively. The latter result settles a problem of Biggs about the footballers of Croam.

1. INTRODUCTION

The graph O_k is a k-valent graph whose vertices are indexed by the $(k-1)$-subsets of a $(2k-1)$-set. Two vertices are adjacent if and only if their indexing subsets are disjoint. The graphs have been studied by Balaban [1, 2] (under the name k-valent halved combination graphs), who encountered them in studying shifts in carbonium ions, and by Biggs [4], who posed the following problem: The eleven footballers of Croam play five a side matches with the eleventh man as referee and each possible choice of teams and referee plays just one match. Is it possible for all 1386 games to be scheduled so that each individual team plays its six games on six different weekdays? He explained that the problem is equivalent to deciding whether O_6 is 6-edge chromatic. (The vertices of O_6 represent the teams and two vertices are joined if and only if the corresponding teams play each other, i.e., if and only if the two teams contain no common players. The six colors represent the weekdays).

A theorem of Vizing [7] states that an n-valent graph G is either n- or $(n+1)$-edge chromatic, and always the latter if G has an odd number of vertices. When k is of the form 2^r the graph O_k does have an odd number of vertices, hence O_2 (a triangle) is 3-edge chromatic and O_4 is 5-edge chromatic. Since O_3 is Petersen's graph it is 4-edge chromatic.

Meredith and Lloyd [5] established that O_4 is an edge disjoint union of two Hamiltonian circuits and that O_5, O_6, O_7 are Hamiltonian. It is well known that Petersen's graph (O_3) is non-Hamiltonian.

DEFINITION. A graph is called m-ply Hamiltonian if it contains m-edge disjoint Hamiltonian circuits.

LEMMA. *If a graph G has an even number of vertices and is m-play Hamiltonian then 2m colors suffice to color the edges in the Hamiltonian circuits.*

Proof. Each Hamiltonian circuit has an even number of edges so can be colored with two colors alternately.

In this paper it is shown that O_5 is doubly Hamiltonian (the remaining edges forming a 1-factor) and that O_6 is triply Hamiltonian (there being no further edges). Both graphs have an even number of vertices; thus O_5 is 5-edge chromatic (the fifth color being required for the 1-factor) and O_6 is 6-edge chromatic. The latter result establishes that the footballers of Croam can schedule their matches so as to avoid Sunday football.

2. SOLUTION OF THE PROBLEM

Since the vertices of the graph O_k are indexed by the subsets of the set $S = \{1, 2,..., 2k - 1\}$, any permutation of S induces a permutation of the vertices of O_k. Also, since two vertices are adjacent if and only if their labeling sets are disjoint, this permuatation is an automorphism of O_k. Thus a group G of permutations of S may be considered as a group of automorphisms of O_k. A quotient graph $Q = O_k/G$ may be constructed as follows: The vertices of Q are indexed by the orbits of vertices of O_k under the action of G. If u is a vertex of O_k then $\bar{u}$ will denote the orbit containing it. The vertices of Q labeled with $\bar{u}$ and $\bar{v}$ are adjacent if and only if there exists $w \in \bar{v}$ such that u and w are adjacent in O_k. Furthermore, the vertices labeled $\bar{u}$ and $\bar{v}$ are joined by r edges if u is adjacent to r different vertices in the orbit $\bar{v}$. Thus, in general Q may have loops and/or multiple edges.

Sands [6] investigated Hamiltonian circuits in O_5 by considering Hamiltonian circuits in the quotient graph $R_5 = O_5/Z_9$ where the cyclic group Z_9 is generated by the cycle (123456789). Meredith and Lloyd [5], using Sands'method, considered $R_6 = O_6/Z_{11}$ and $R_7 = O_7/Z_{13}$. If O_k is reduced by a group larger than Z_{2k-1} then the orbits of vertices are generally of different sizes. If O_k is reduced by Z_{2k-1} the orbits are all of the same size but the quotient graph contains loops since, for example, the vertices

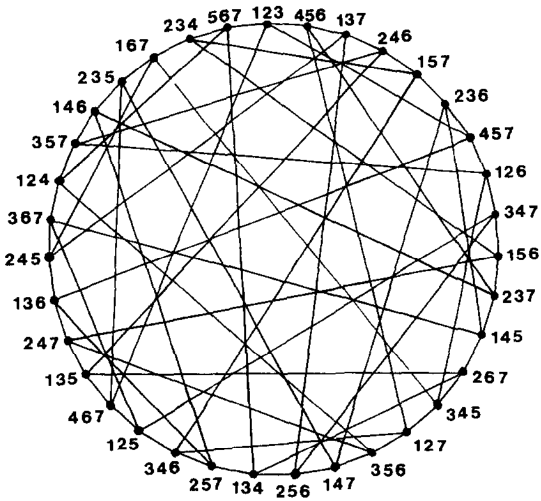

FIG. 1. The graph O_4 as an edge disjoint union of two Hamiltonian circuits with five-fold symmetry.

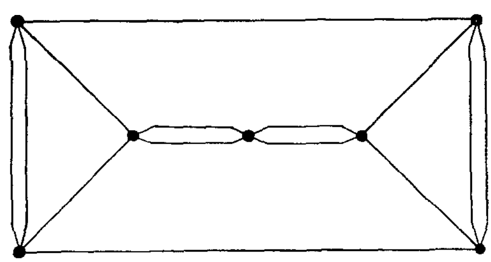

FIG. 2. The graph $O_4/(45763)$.

labeled $\{1, 2,..., k - 1\}$ and $\{k + 1, k + 2,..., 2k - 1\}$ are adjacent and in the same orbit. The results of the present paper were obtained by reducing by a group smaller than Z_{2k-1}.

Balaban [3] showed that O_4 is a union of two Hamiltonian circuits with five-fold symmetry by drawing it as in Figure 1. The symmetry is induced by the group generated by the cycle (45763). The quotient graph is shown in Figure 2. Clearly this reduced graph is doubly Hamiltonian. There are 32 different Hamiltonian circuits splitting up into 16 pairs of edge disjoint circuits. In eight cases the Hamiltonian circuits lift back to Hamiltonian circuits in O_5, in the other eight to a Hamiltonian circuit and five circuits of length seven. Some care is needed when checking this to ensure that both edges of a double edge are lifted back, and not the same edge twice.

It is now natural to reduce O_5 by (1234567). The resulting graph is shown, in two parts, in Figure 3. It is also without loops, doubly Hamiltonian, and a suitable choice of circuits leads to two Hamiltonian circuits in O_5.

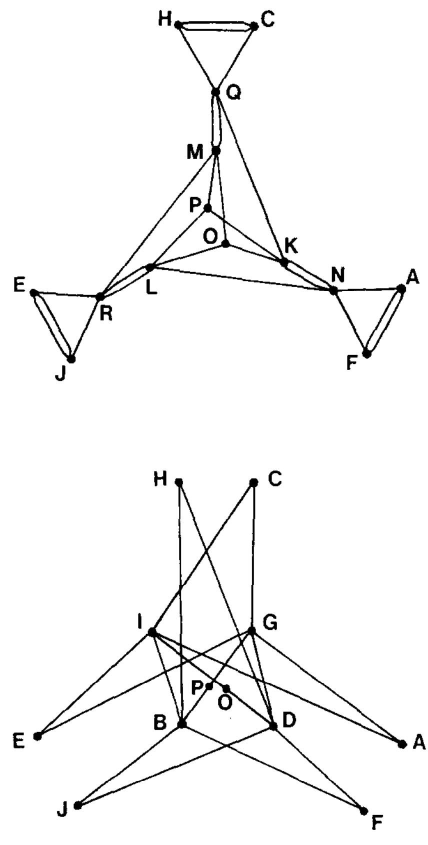

FIG. 3. The graph $O_5/(1234567)$.

The labeling sets of two adjacent vertices in O_k exhaust $2k - 2$ of the elements $1,..., 2k - 1$ and so the edge joining them can be labeled with the "missing" element. To specify a Hamiltonian circuit in O_k uniquely it suffices (and saves space) to name an initial vertex and then to list the edge labels in order. This method is used in Table 1 to list two edge disjoint Hamiltonian circuits in O_5. The first circuit, for example, 1234 : 6 4 9... corresponds to the circuit through vertices 1234, 5789, 1236, 4578,...

Since nine is not prime the orbits of O_6 reduced by (123456789) are of

TABLE 1

Two Edge Disjoint Hamiltonian Circuits in O_5

1234:	6 4 9 1 8 3 4 8 1 6 5 1 3 5 6 9 5 6
	7 5 9 2 8 4 5 8 2 7 6 2 4 6 7 9 6 7
	1 6 9 3 8 5 6 8 3 1 7 3 5 7 1 9 7 1
	2 7 9 4 8 6 7 8 4 2 1 4 6 1 2 9 1 2
	3 1 9 5 8 7 1 8 5 3 2 5 7 2 3 9 2 3
	4 2 9 6 8 1 2 8 6 4 3 6 1 3 4 9 3 4
	5 3 9 7 8 2 3 8 7 5 4 7 2 4 5 9 4 5
1234:	8 4 6 8 5 6 7 2 9 3 2 1 6 7 3 9 4 2
	8 6 1 8 7 1 2 4 9 5 4 3 1 2 5 9 6 4
	8 1 3 8 2 3 4 6 9 7 6 5 3 4 7 9 1 6
	8 3 5 8 4 5 6 1 9 2 1 7 5 6 2 9 3 1
	8 5 7 8 6 7 1 3 9 4 3 2 7 1 4 9 5 3
	8 7 2 8 1 2 3 5 9 6 5 4 2 3 6 9 7 5
	8 2 4 8 3 4 5 7 9 1 7 6 4 5 1 9 2 7

TABLE 2

Three Edge Disjoint Hamiltonian Circuits in O_6
(each commences at vertex 12345 and X denotes 10 and E denotes 11)

I	E 2 8 E 2 5 6 1 E 8 1 3 5 4 8 1 X 6 7 E 4 X E 4
	9 E 3 7 4 3 6 2 1 6 7 9 3 8 X 3 8 4 6 7 4 6 3 1
	9 8 6 9 2 3 9 X 3 6 7 2 X 9 2 X 6 2 ....
	by applying (1357246).
II	6 4 E 5 9 E 5 3 7 1 E 9 1 2 3 6 9 7 8 5 2 E 7 8
	E 3 X E 6 1 3 6 4 2 5 4 2 X 1 9 8 1 9 5 6 3 5 6
	1 7 X 9 6 X 3 5 X 8 5 3 7 1 8 X 1 8 ....
	by applying (1526374).
III	9 1 X 3 7 2 3 7 1 4 8 X 7 8 2 3 8 9 3 2 4 1 9 8
	1 9 2 1 E 5 X E 5 4 6 3 E X 3 7 4 2 X 3 9 6 1 E
	7 9 E 7 8 E 7 1 2 7 6 5 3 6 5 8 6 X ....
	by applying (1234567).

varying sizes and the above technique does not solve the footballers problem. Reducing by (1234567), however, leads to a loopless graph with 66 vertices which is triply Hamiltonian. It is too big to draw here but a suitable choice of Hamiltonian circuits leads to Hamiltonian circuits in O_6 which are listed in Table 2. It is, of course, only necessary to list the first 66 edge labels since the remainder can be obtained by cyclic permutation. (Cf. Table 1, where the first 18 edge labels and the cycle (1234567) determine the remainder of the first circuit.)

CONJECTURE. *The graphs O_{2k} and O_{2k+1} are k-ply Hamiltonian for all* $k \geqslant 2$.

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